MATH 2028 Honours Advanced Calculus II
2023-24 Term 1
Problem Set 7
due on Nov 17, 2023 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: All surfaces are contained inside $\mathbb{R}^3$ with rectangular coordinates (x, y, z). We use U to denote a bounded open subset of $\mathbb{R}^2$.

Problems to hand in

1. Let $a > 0$ be a fixed constant. Find the area of the portion of the cylinder $x^2 + y^2 = a^2$ lying above the xy-plane and below the plane $z = y$.
2. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$. Calculate $\int_S x^2 \, d\sigma$. *(Hint: make use of the symmetry)*
3. Let S be the portion of the plane $x + 2y + 2z = 4$ lying in the first octant of $\mathbb{R}^3$, oriented with outward normal pointing upward. Find
   (a) the area of S,
   (b) $\int_S (x - y + 3z) \, d\sigma$,
   (c) $\int_S F \cdot \vec{n} \, d\sigma$ where $F(x, y, z) = (x, y, z)$.
4. Let S be the portion of the helicoid given by the parametrization $g(u, v) : (0, 1) \times (0, 2\pi) \to \mathbb{R}$ by
$$g(u, v) = (u \cos v, u \sin v, v).$$
Suppose S is oriented by the upward pointing unit normal $\vec{n}$. Compute $\int_S F \cdot \vec{n} \, d\sigma$ where $F(x, y, z) = (0, x, 0)$.

Suggested Exercises

1. Find the area of the portion of the cone $z = \sqrt{2(x^2 + y^2)}$ lying beneath the plane $y + z = 1$.
2. Find the area of the portion of the cylinder $x^2 + y^2 = 2y$ lying inside the sphere $x^2 + y^2 + z^2 = 4$.
3. Find the flux of the vector field $F(x, y, z) = (x^2, y^2, z^2)$ outward across the given surface S (all oriented with outward pointing normal pointing away from the origin, unless otherwise specified):
   (a) S is the sphere of radius a centered at the origin.
   (b) S is the upper hemisphere of radius a centered at the origin.
   (c) S is the cone $z = \sqrt{x^2 + y^2}$, $0 < z < 1$, with outward pointing normal having a negative z-component.
   (d) S is the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$.

(e) S is the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, along with the disks $x^2 + y^2 \le a^2$, $z = 0$ and $z = h$.

4. Calculate the flux of the vector field $F(x, y, z) = (xz, yz, x^2 + y^2)$ outward across the surface of the paraboloid S given by $z = 4 - x^2 - y^2$, $z \ge 0$ (with outward pointing normal having positive z-component).

5. Compute $\int_S F \cdot \vec{n}\, d\sigma$ where $F(x, y, z) = (x, y, z)$ for each of the following surfaces in $\mathbb{R}^3$ (all oriented with the outward pointing unit normal pointing away from the origin):

   (a) the sphere of radius a centered at the origin,

   (b) the cylinder $x^2 + y^2 = a^2$, $-h \le z \le h$,

   (c) the cylinder $x^2 + y^2 = a^2$, $-h \le z \le h$, together with the two disks $x^2 + y^2 \le a^2$, $z = \pm h$,

   (d) the cube with vertices at $(\pm 1, \pm 1, \pm 1)$.

6. Repeat the question above for the vector field $F(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}(x, y, z)$.

7. Prove that the area of a graphical surface S given by $z = f(x, y)$, where $f : U \to \mathbb{R}$ is a C^1 function, is given by

$$\text{Area}(S) = \iint_U \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

Challenging Exercises

1. Prove or give a counterexample: If S is an orientable surface, then there are exactly two possible orientations on S.

2. Let $\alpha, \beta, f : [0, 1] \to \mathbb{R}$ be C^1 functions with $f(t) > 0$ for all $t \in [0, 1]$. Suppose that S is a surface in $\mathbb{R}^3$ whose intersection with the plane $z = t$ is the circle

$$(x - \alpha(t))^2 + (y - \beta(t))^2 = (f(t))^2, \qquad z = t$$

   for each $t \in [0, 1]$ and is empty for $t \notin [0, 1]$.

   (a) Set up an integral for the area of S.

   (b) Evaluate the integral in (a) when α and β are constant functions and $f(t) = (1 + t)^{1/2}$.

   (c) What form does the integral take when f is constant and $\alpha(t) = 0$ and $\beta(t) = at$ where a is a constant?